* Principal of General Equation of second degree

Art 1

Find eqns to the principal planes of the conicoid

$F(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$

$\Rightarrow$ The given conicoid is

$$F(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \quad \text{—— (1)}$$

The diametral plane bisecting chords parallel to

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{—— (2)}$$

is $x(al + hm + gn) + y(hl + bm + fn) + z(gl + fm + cn) + (ul + vm + wm) = 0$ —— (3)

If (3) is $\perp$ to (2)

$$\frac{al + hm + gn}{l} = \frac{hl + bm + fn}{m} = \frac{gl + fm + cn}{n} = \lambda \text{ (say)} \quad \text{—— (4)}$$

$$\frac{al + hm + gn}{l} = \lambda$$

$$al + hm + gn = \lambda l$$

$$(a-\lambda)\begin{vmatrix} (b-\lambda) & f \\ f & (c-\lambda) \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c-\lambda \end{vmatrix} + g\begin{vmatrix} h & b-\lambda \\ g & f \end{vmatrix} = 0$$

$$(a-\lambda)\left[(b-\lambda)(c-\lambda) - f^2\right] - h\left[h(c-\lambda) - fg\right] + g\left[hf - g(b-\lambda)\right] = 0$$

$$(a-\lambda)\left[bc - b\lambda - \lambda c + \lambda^2 - f^2\right] - h\left[hc - h\lambda - fg\right] + g\left[hf - gb + g\lambda\right] = 0$$

$$abc - ab\lambda - a\lambda c + a\lambda^2 - af^2 - h^2c + h^2\lambda + hfg + ghf - g^2b + g^2\lambda$$
$$- \lambda bc + \lambda^2 b + \lambda^2 c - \lambda^3 + \lambda f^2 = 0$$

$$-\lambda^3 + \lambda^2(a+b+c) - \lambda(ab+bc+ac - f^2 - g^2 - h^2)$$
$$+ (abc + 2hgf - af^2 - bg^2 - ch^2) = 0$$

$$\lambda^3 - \lambda^2(a+b+c) + \lambda(ab+bc+ac - f^2 - g^2 - h^2)$$
$$- (abc + 2hgf - af^2 - bg^2 - ch^2) = 0 \quad \text{--- (8)}$$

$$\lambda^3 - \lambda^2(a+b+c) + \lambda(A+B+C) - D = 0 \quad \text{--- (9)}$$

eq^n (8) is called Discriminating cubic

The eq^n has 3 roots each of which corresponds one principal direction l, m, n

Substituting the values of $l : m : n$ in (3) we get

$$\lambda(lx + my + nz) + ul + vm + wn = 0$$

Ex-1 Reduce the equation.

$$11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy + 72x - 72y + 36z + 150 = 0$$

to standard form and show that it represents an ellipsoid and find the eq^n of the axes.

→ The given eq^n is

$$11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy + 72x - 72y + 36z + 150 = 0$$

Comparing it with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

we get

$a = 11$ $\quad f = -4$ $\quad u = 36$ $\quad d = 150$

$b = 10$ $\quad g = 2$ $\quad V = -36$

$c = 6$ $\quad h = -6$ $\quad w = 18$

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 11 & -6 & 2 \\ -6 & 10 & -4 \\ 2 & -4 & 6 \end{vmatrix}$$

$$= 11 \begin{vmatrix} 10 & -4 \\ -4 & 6 \end{vmatrix} + 6 \begin{vmatrix} -6 & -4 \\ 2 & 6 \end{vmatrix} + 2 \begin{vmatrix} -6 & 10 \\ 2 & -4 \end{vmatrix}$$

$$= 11[60 - 16] + 6[-36 + 8] + 2[24 - 20]$$

$$= 11(44) + 6(-28) + 2(4)$$

$$= 484 - 168 + 8 = 324 \neq 0$$

The discriminating cubic is

$$\begin{vmatrix} 11-\lambda & -6 & 2 \\ -6 & 10-\lambda & -4 \\ 2 & -4 & 6-\lambda \end{vmatrix} = 0$$

$$(11-\lambda) \begin{vmatrix} (10-\lambda) & -4 \\ -4 & (6-\lambda) \end{vmatrix} + 6 \begin{vmatrix} -6 & -4 \\ 2 & (6-\lambda) \end{vmatrix} + 2 \begin{vmatrix} -6 & (10-\lambda) \\ 2 & -4 \end{vmatrix} = 0$$

$$(11-\lambda)\left[(10-\lambda)(6-\lambda) - 16\right] + 6\left[-6(6-\lambda) + 8\right] + 2\left[24 - 2(10-\lambda)\right] = 0$$

$$-\lambda^3 + 27\lambda^2 - 220\lambda + 484 + 36\lambda - 168 + 4\lambda + 8 = 0$$

$$-\lambda^3 + 27\lambda^2 - 180\lambda + 324 = 0$$

$$\lambda^3 - 27\lambda^2 + 180\lambda - 324 = 0$$

$$(\lambda-3)(\lambda-6)(\lambda-18) = 0$$

$$\lambda = 3, 6, 18$$

The Central planes are

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial z} = 0$$

$$11x - 6y + 2z + 36 = 0 \quad \text{—} \; (1)$$

$$3x - 5y + 2z + 18 = 0 \quad \text{—} \; (2)$$

$$x - 2y + 3z + 9 = 0 \quad \text{—} \; (3)$$

Subtracting (3) from (1)

$$8x - y + 18 = 0 \quad \text{—} \; (4)$$

Multiplying (2) x 3 and (3) by 2 we get

$$9x - 15y + 6z + 54 = 0 \quad \text{—} \; (5)$$

$$2x - 4y + 6z + 18 = 0 \quad \text{—} \; (6)$$

Subtracting (6) from (5)

we get

$$7x - 11y + 36 = 0 \quad \text{—— (7)}$$

Solving (4) and (7)

$$\frac{x}{-36+198} = \frac{y}{126-288} = \frac{1}{-88+7}$$

$$\frac{x}{162} = \frac{y}{-162} = \frac{1}{-81}$$

$$x = \frac{162}{-81} = -2$$

$$y = \frac{-162}{-81} = 2$$

from (3) $-2-4+3z+9=0$

$$3z = -3$$

$$z = -1$$

Centre is $(-2, 2, -1)$

$\alpha = -2$, $\beta = 2$, $\gamma = -1$

$$d' = u\alpha + v\beta + w\gamma + d$$

$$= 36(-2) - 36(2) + 18(-1) + 150$$

$= -72 - 72 - 18 + 150 = -12$

The eq^n of conicoid referred to the centre $(-2, 2, -1)$ as origin is

$$11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy - 12 = 0$$

On suitable rotation of axes, the eq^n reduces to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$$

$$3x^2 + 6y^2 + 18z^2 - 12 = 0$$

$$\frac{x^2}{4} + \frac{y^2}{2} + \frac{z^2}{\frac{2}{3}} = 1$$

which is ellipsoid.

The principal direction is given by

$$(a-\lambda)l + hm + gn = 0$$

$$hl + (b-\lambda)m + fn = 0$$

$$gl + fm + (c-\lambda)n = 0$$

from first two eq^{n}s we have

$$\frac{l}{fh - g(b-\lambda)} = \frac{m}{gh - f(a-\lambda)} = \frac{n}{(a-\lambda)(b-\lambda) - h^2}$$

When $\lambda = 3$

$$\frac{l}{(-4)(-6)-2(10-3)} = \frac{m}{(2)(-6)-(-4)(11-3)} = \frac{n}{(11-3)(10-3)-(-6)^2}$$

$$\frac{l}{24-14} = \frac{m}{-12+32} = \frac{n}{56-36}$$

$$\frac{l}{10} = \frac{m}{20} = \frac{n}{20}$$

$$\frac{l}{1} = \frac{m}{2} = \frac{n}{2}$$

When $\lambda = 6$ we have

$$\frac{l}{24-12} = \frac{m}{-12+20} = \frac{n}{20-36}$$

$$\frac{l}{16} = \frac{m}{8} = \frac{n}{-16}$$

$$\frac{l}{2} = \frac{m}{1} = \frac{n}{-2}$$

when $\lambda = 18$

$$\frac{l}{24-2(-8)} = \frac{m}{-12+4(-7)} = \frac{n}{(-7)(-8)-36}$$

$$\frac{l}{40} = \frac{m}{-40} = \frac{n}{20}$$

$$\frac{l}{2} = \frac{m}{-2} = \frac{n}{1}$$

Hence the axes are

$$\frac{x+2}{1} = \frac{y-2}{2} = \frac{z+1}{2}$$

$$\frac{x+2}{2} = \frac{y-2}{1} = \frac{z+1}{-2}$$

$$\frac{x+2}{2} = \frac{y-2}{-2} = \frac{z+1}{1}$$